L.O. LIMITED FREQUENCY STABILITY FOR PASSIVE ATOMIC FREQUENCY STANDARDS USING "SQUARE WAVE" FREQUENCY MODULATION*

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Abstract

Atomic frequency standards using square-wave frequency modulation effectively interrogate the atomic line by switching back and forth between two frequencies with equal atomic absorption values. For a symmetric absorption line, the slope of the responses will also be equal. In the quasi-static limit this would seem to be an ideal interrogation process – the sign reversal of frequency slope can be removed by detection electronics to give an essentially unvarying and constant sensitivity to L.O. frequency variations. Such an interrogation would seem to eliminate L.O. aliasing and so relieve stringent requirements on local oscillator phase noise. However, sign changes in the interrogation and detection processes mean that the sensitivity is actually zero at some point in the cycle. We derive consequences of this fact for white phase and flicker phase noise models by an explicitly time-dependent analysis in terms of the sensitivity function $g(t)$. We find that an optimal strategy exists for white phase L.O. noise for which $g(t)$ takes the form of a sequence of parabolic arches, and which gives a small ($\times 96/\pi^4$) improvement over simple sine-wave demodulation. We also find limiting forms that could in principle eliminate L.O. aliasing for flicker-phase noise. However, in practice the improvement shows only a logarithmic dependence on available response time and bandwidth.

1 Introduction

Limitations to passive frequency performance by aliased L.O. fluctuations have been previously analyzed for both pulse- and cw- interrogation methods. Up to now, frequency standards using cw frequency modulation interrogation, such as Cesium beam tubes and Rubidium gas cells, have been analyzed using a frequency (Fourier) approach, while a time-dependent approach has been applied to pulse-mode standards such as the Linear Ion Trap and Cesium Fountain[1][2]. We present here an application of the time-dependent methodology to a limiting case of square wave frequency modulation for passive frequency standards operating in a nominally "cw" mode and treated in the quasi-static limit.

The effectiveness of square wave demodulation has been shown in recent work to be surprisingly smaller than might be expected. Barillet et al [2] find only minimal ($\approx 1\%$) performance improvement by including successively higher numbers of optimized harmonics in the detection waveform for the case of white phase noise. The surprise is that, mathematically, a constant sensitivity can be approached by increasing harmonic content, and an unvarying sensitivity would not cause any aliasing at all.

On the other hand, the actual limiting value of aliasing for square wave frequency modulation and detection may be different from zero. Since both the time-varying sensitivity of the atomic discriminator and the transfer function of the modulator must in fact cross the axis while reversing sign, it makes sense to consider a general symmetric form of the sensitivity $g(t)$ that goes from 0 to 1 and back to 0 during half a cycle of the square wave, and to study the limiting behavior of the aliasing as $g(t)$ approaches a rectangle. For this form that always includes zeros, it is not obvious that aliasing is small.

We analyze limiting cases for the problematical white and flicker phase noise processes in terms of a dimensionless parameter δ that describes the deviation of $g(t)$ from a rectangle. Working from explicit time dependencies rather than harmonic content, we calculate the spectral density of aliased frequency variations $S_y^{\mathrm{LLO}}(0)$ for the locked local oscillator. Normalizing to the case of sine wave modulation and demodulation, we prove that aliasing diverges as $1/\delta$ for white phase noise for

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all choices of $g(t)$. For white phase noise we also derive the optimal shape of $g(t)$, a parabolic arch that reduces aliasing from our nominal case by a factor of $6/\pi^2$. Its harmonic content agrees with [2]. However, aliasing for the optimal case is only 1.5% smaller than for sine modulation with square wave demodulation ($\pi^2/16$), explaining why improvements to that widely used strategy have proven so hard to come by. Finally, we present forms for $g(t)$ for which the aliasing of flicker phase noise approaches zero with decreasing δ.

2 Approach

In the quasi-static limit, the sensitivity function $g(t)$ is given by the common sense definition as a product of two terms. The first is the time varying sensitivity given by the slope of the atomic response with frequency. The second is the multiplicative demodulation waveform. Square wave modulation, as opposed to demodulation, is typically considered because otherwise the nonlinearity of the atomic response with frequency must be factored in.

It may not be immediately obvious what the relationship of such a a time-dependent approach is to a physical situation where demodulation occurs only after the Fourier components of the waveform are filtered or selected. Conceptually, the solution is to replace any such sine wave filter by a corresponding multiplication by a sine wave. It may be argued that because the filter in fact does not select a phase, it does something somehow different from that of a sine wave demodulator. This is not the case, however, since both the square wave modulator and the following demodulator together completely constrain the phase of the sine wave components that contribute to a demodulated signal.

3 Sensitivity and aliasing

Let the modulation and demodulation waveforms be denoted by $s_1(t)$ and $s_2(t)$, respectively. Both waveforms are assumed to be periodic with period $T_m = 1/f_m$, odd about $t = 0$, and even about $t = T_m/4$. Then in the quasistatic limit the sensitivity function is $g(t) = s_1(t)\, s_2(t)$, and is periodic with period $T_c = T_m/2$ and even about $t = 0$ and $T_c/2$. We assume that $g(0) = 0$, and that $g(t)$ is continuous and increases monotonically for $0 \le t \le T_c/2$.

For a close study of the shape of $g(t)$, it is convenient to define a scaled, reflected version, called the reduced

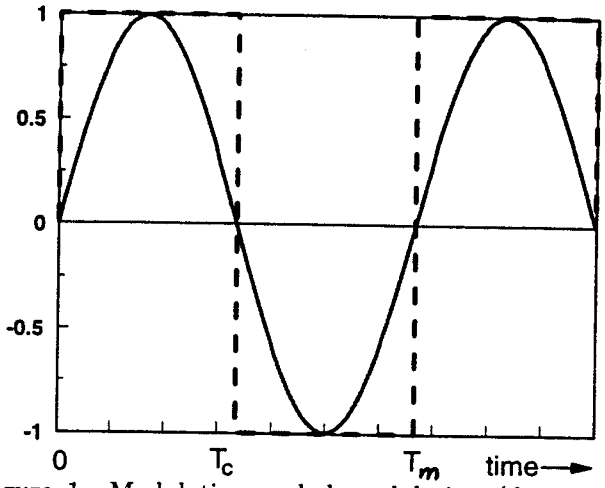

Figure 1: Modulation and demodulation (detection) waveforms typically considered for FM interrogation in the quasi-static limit. The dashed square wave modulation curve is indicated by $s_1(t)$ in the text and the smoother demodulation waveform by $s_2(t)$.

sensitivity, by

$$g_r(u) = 1 - \frac{g\left(\frac{1}{2}T_c u\right)}{g\left(\frac{1}{2}T_c\right)}.$$

Then $g_r(0) = 1$, $g_r(1) = 0$, and $g_r(u)$ decreases monotonically for $0 \le u \le 1$. A natural dimensionless parameter that measures how much $g(t)$ differs from a rectangle can be defined by

$$\delta = \int_0^1 g_r(u)\, du.$$

In the present setup, the Dick formula [1] for the noise level of the aliased white FM takes the form

$$S_y^{\mathrm{LLO}}(0) = \frac{2}{(1-\delta)^2} \sum_{n=1}^{\infty} g_n^2 S_y^{\mathrm{LO}}(nf_c), \qquad (1)$$

where the g_n are now defined by

$$g_n = \int_0^1 g_r(u) \cos(\pi n u)\, du, \quad n \ge 1.$$

The Fourier series of $g_r(u)$ as an even function on $[-1, 1]$ is given by

$$g_r(u) = \delta + 2\sum_{n=1}^{\infty} g_n \cos(\pi n u).$$

Since $g_r(0) = 1$ we have

$$1 - \delta = 2\sum_{n=1}^{\infty} g_n. \qquad (2)$$

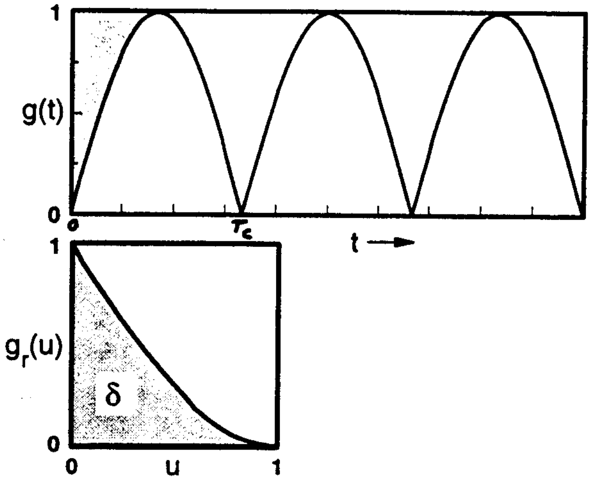

Figure 2: The sensitivity function $g(t)$ for square-wave modulation and sine-wave detection or demodulation. Such a symmetric waveform can be conveniently represented by the reduced form $g_r(t)$ in the lower plot.

3.1 Reference case

This is the case of sine-wave modulation and demodulation. If $s_1(t) = s_2(t) = \sin(2\pi f_m t)$, then $g_r(u) = \frac{1}{2}(1+\cos \pi u)$, from which $\delta = \frac{1}{2}$, $g_1 = \frac{1}{4}$, and the other g_n vanish. The resulting Dick formula is written as

$$S_y^{\mathrm{LLO}}\left(0; \sin^2\right) = \frac{1}{2} S_y^{\mathrm{LO}}\left(f_c\right). \qquad (3)$$

To give a convenient dimensionless presentation of the aliasing effect, we normalize (1) by (3), giving

$$A = \frac{4}{(1-\delta)^2} \sum_{n=1}^{\infty} g_n^2 \frac{S_y^{\mathrm{LO}}\left(n f_c\right)}{S_y^{\mathrm{LO}}\left(f_c\right)} \qquad (4)$$

as the factor that compares the aliasing for the general case to the aliasing for sine-sine detection. The expression A is 4 times the expression ξ^c of Barillet et al. [2] provided that we make the identification

$$n^2 \lambda_{2n} = \frac{S_y^{\mathrm{LO}}\left(n f_c\right)}{S_y^{\mathrm{LO}}\left(f_c\right)}.$$

Our purpose is to study the behavior of A as a function of $g_r(u)$ and $S_y^{\mathrm{LO}}(f)$. For power-law LO noise, $S_y^{\mathrm{LO}}(f) = h_\alpha f^\alpha$, we write $A = A_\alpha$, where

$$A_\alpha = \frac{4}{(1-\delta)^2} \sum_{n=1}^{\infty} g_n^2 n^\alpha. \qquad (5)$$

The cases $\alpha = 2$ (white PM) and $\alpha = 1$ (flicker PM) are examined below.

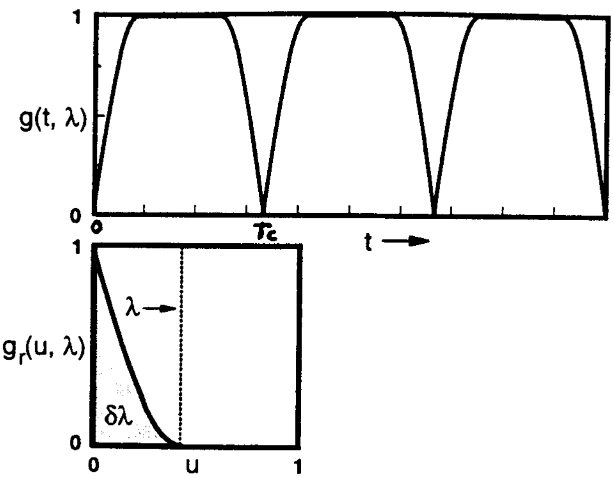

Figure 3: Reducing δ by horizontally shrinking a chosen shape is one way to approach a rectangle.

3.2 Horizontal shrinking

In studying how the aliasing effect A depends on the closeness of the sensitivity function to a rectangle, one approach is to start with a fixed $g_r(u)$ and shrink it horizontally as in Fig. 3: for $0 < \lambda \le 1$ let $g_r(u, \lambda) = g_r(u/\lambda)$ for $0 \le u \le \lambda$, and $g_r(u, \lambda) = 0$ for $\lambda < u \le 1$. Then everything becomes a function of λ, so that $\delta(\lambda) = \lambda \delta(1)$, $g_n(\lambda) = \lambda G(n\lambda)$, where

$$G(x) = \int_0^1 g_r(u) \cos(\pi x u)\, du.$$

From (5) we have

$$A_\alpha(\lambda) = \frac{4\lambda^{1-\alpha}}{[1-\lambda\delta(1)]^2} \sum_{n=1}^{\infty} G^2(n\lambda)(n\lambda)^\alpha \lambda. \qquad (6)$$

Under reasonable conditions on $g_r(u)$ it can be shown that the infinite Riemann sum in (6) converges to $\int_0^\infty G^2(x) x^\alpha dx$ as $\lambda \to 0$; consequently, $A_\alpha(\lambda)$ is asymptotic to a constant (depending on the original $g_r(u)$) times $\lambda^{1-\alpha}$. For white PM, $A_2(\lambda)$ diverges like $1/\lambda$; for flicker PM, $A_1(\lambda)$ tends to a positive finite limit.

These results lead to conjectures for the general case in which δ replaces λ as the measure of closeness. The conjecture that A_2 is greater than some positive constant times $1/\delta$ is true, and is proved below. The corresponding conjecture that A_1 is greater than some positive constant is false; counterexamples are given below.

4 White PM

Here it is convenient to introduce the derivative $w(u) = -g_r'(u)$, a unit weight function on $[0, 1]$, i.e., $w(u) \ge 0$

and $\int_0^1 w(u)\,du = 1$. Then $\delta = \int_0^1 uw(u)\,du$, the center of mass, and $g_n = w_n/(\pi n)$, where

$$w_n = \int_0^1 w(u)\sin(\pi n u)\,du, \quad n \geq 1. \tag{7}$$

From (5) we have

$$A_2 = \frac{4}{\pi^2(1-\delta)^2}\sum_{n=1}^{\infty} w_n^2, \tag{8}$$

and, because the functions $\sqrt{2}\sin(\pi n u)$ form a complete orthonormal system for $0 \leq u \leq 1$, we also have

$$A_2 = \frac{2}{\pi^2(1-\delta)^2}\int_0^1 w^2(u)\,du. \tag{9}$$

4.1 Optimal detection

It is natural to ask what form of sensitivity function is optimal for white PM. This problem has an easy solution. Let $m(u) = 1-u$; then $1-\delta = \int_0^1 m(u)\,w(u)\,du$, and (9) can be expressed as

$$\frac{1}{A_2} = \frac{\pi^2}{2}\left\langle \frac{w}{\|w\|}, m\right\rangle^2,$$

where $\langle\, ,\rangle$ and $\|\cdot\|$ represent the inner product and norm in $L^2(0,1)$. This quantity attains its maximum when the unit vector $w/\|w\|$ points in the same direction as m. Since $\int_0^1 w(u)\,du = 1$, we have $w(u) = 2(1-u)$, $g_r(u) = (1-u)^2$, and $\min A_2 = 6/\pi^2$, attained by square-wave modulation and a demodulation waveform consisting of a series of positive and negative parabolic arches.

This result confirms that of Barillet et al. [2], who proceed by expanding the demodulation waveform $s_2(t)$ as a sine series; their optimal coefficients $C_{2k+1} = (2k+1)^{-3}$ are indeed the sine coefficients of a parabolic-arch wave.

Yet another path to the same result proceeds by expressing A_2 in terms of w_n only: from (8) and (2) we have

$$A_2 = \frac{\sum_{n=1}^{\infty} w_n^2}{\left(\sum_{n=1}^{\infty} w_n n^{-1}\right)^2}. \tag{10}$$

By the same argument as before, but in the space l^2 of square-summable sequences, the optimal w_n is proportional to $1/n$, which in turn is proportional to the nth sine coefficient of $1-u$.

4.2 Square-sine detection

Because a sine wave looks like a parabolic-arch wave (Fig. 4), it is interesting to compute aliasing of white PM for square-wave modulation followed by sine-wave demodulation. A straightforward evaluation of (9) with $g_r(u) = 1 - \sin\left(\frac{1}{2}\pi u\right)$ gives $A_2 = \pi^2/16$, which is greater than the minimal A_2 by a factor of $\pi^4/96 = 1.0147$.

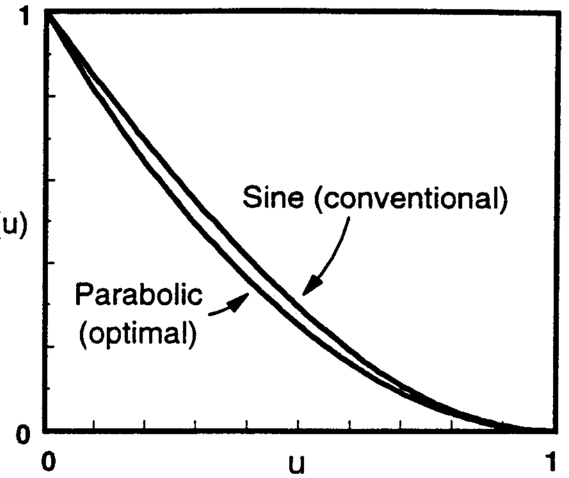

Figure 4: Comparing the reduced sensitivities for sine and parabolic demodulation.

4.3 Divergence of aliasing

The resolution of the divergence conjecture for white PM is contained in the following assertion:
if $\delta \leq 1/3$ then

$$A_2 \geq \frac{8}{27\pi^2}\frac{1}{\delta(1-\delta)^2}. \tag{11}$$

To prove this from (9) we seek a lower bound for $\int_0^1 w^2(u)\,du$ given $w(u) \geq 0$, $\int_0^1 w(u)\,du = 1$, $\int_0^1 uw(u)\,du = \delta$. Fix $r > 1$ and let $\delta \leq 1/r$. Then $\delta \geq \int_{r\delta}^1 uw(u)\,du \geq r\delta \int_{r\delta}^1 w(u)\,du$; hence $\int_{r\delta}^1 w(u)\,du \leq 1/r$, $\int_0^{r\delta} w(u)\,du \geq 1 - 1/r$. In other words, $w(u)$ has appreciable mass in the interval $[0, r\delta]$. Now, by Schwarz's inequality, $\left(\int_0^{r\delta} w(u)\,du\right)^2 \leq r\delta \int_0^{r\delta} w^2(u)\,du$, whence

$$\int_0^1 w^2(u)\,du \geq \int_0^{r\delta} w^2(u)\,du \geq \frac{1}{r\delta}\left(\int_0^{r\delta} w(u)\,du\right)^2 \geq \frac{1}{r\delta}\left(1 - \frac{1}{r}\right)^2.$$

The best choice of r for this purpose is 3.

Although this proposition does prove the conjecture, it is clear that the constant in (11) is far from being the best possible. To improve it, sharper mathematical tools than Chebyshev's and Schwarz's inequalities will be needed.

5 Flicker PM

The issue here is the conjecture that there exists a universal positive constant C such that $A_1 \geq C$. With the right viewpoint it is not hard to show that this conjecture is false. Let $x_n = g_n n^{1/2}$. From (5) and (2) we have

$$A_1 = \frac{\sum_{n=1}^{\infty} x_n^2}{\left(\sum_{n=1}^{\infty} x_n n^{-1/2}\right)^2}. \tag{12}$$

We cannot proceed quite in the same way as with (10) above because $n^{-1/2}$ is not square-summable. Instead, we fix a positive integer N and assume that x_n vanishes for $n > N$. Now we can write $A_1^{-1} = \langle x/\|x\|, p\rangle^2$, where $p_n = n^{-1/2}$, $n = 1, \ldots, N$, and the inner product and norm are those of Euclidean N-space. The maximum A_1^{-1} is achieved when x_n is proportional to p_n, that is, when g_n is proportional to $1/n$ for $1 \leq n \leq N$ and zero otherwise. This solution has to be modified slightly to give a monotonic $g_r(u)$. Let

$$f(u) = \sum_{n=1}^{N-1} \frac{\cos(\pi n u)}{n} + \frac{\cos(\pi N u)}{2N}.$$

By differentiating $f(u)$ we can show that $f(u)$ decreases for $0 \leq u \leq 1$. Set $g_r(u) = \delta + 2b f(u)$, and assign to δ and b the values that make $g_r(0) = 1$, $g_r(1) = 0$. Knowing that $\sum_{n=1}^{N} 1/n = \ln N + \gamma + o(1)$, where $\gamma =$ Euler's constant, we find that δ and A_1 both tend to zero like $1/\ln N$ as $N \to \infty$. It is impossible to make A_1 tend to zero any faster than that, so that, although the conjecture is false in a mathematical sense, it remains true in a practical sense. Fig. 5 shows $g_r(u)$ for $N = 200$.

Another example is given by the simple formula

$$g_r(u) = 1 - \frac{\ln(1 + au)}{\ln(1 + a)},$$

where a is a positive parameter that is allowed to tend to infinity. With some effort it can be shown that δ and A_1 both tend to zero like $1/\ln a$. The shape of this function is like that of the previous example: a sharp point on a sturdy base.

6 Conclusions

We have quantitatively examined the attractive and almost obvious idea that as a "constant" interrogation is approached, L.O. aliasing must go away. This idea turns out not to be true because the the sensitivity must actually be zero at some point. For noise models that have a substantial high frequency component, having an almost constant sensitivity means having less time left to get to the zero point, and that necessarily picks out high frequency noise components.

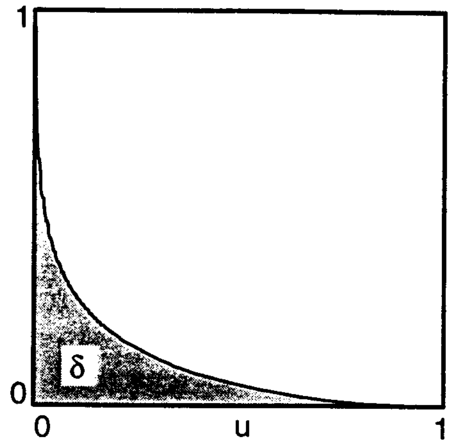

Figure 5: A member ($N = 200$) of a family of reduced sensitivity functions $g_r(u)$ for which aliasing tends to zero like $1/\ln N$ in the presence of flicker PM LO noise.

The atomic response is a real number of counts per second and does not have an imaginary conjugate that can also be observed. So there seems no way around the problem.

References

[1] G. J. Dick, "Local oscillator induced instabilities in trapped ion frequency standards", *Proc. 19th Precise Time and Time Interval (PTTI) Applications and Planning Meeting*, pp. 133–147, Redondo Beach, CA, 1987.

[2] R. Barillet, D. Venot, and C. Audoin, "Reducing the effect of interrogation phase noise on the frequency stability in passive frequency standards: experimental results and new investigation means", *Proc. 1995 International Frequency Control Symposium*, pp. 178–184, 1995.